# Year 12 Mathematics

# IAS 2.14

## Systems of Equations

Robert Lakeland & Carl Nugent

## Contents



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## NCEA 2 Internal Achievement Standard 2.14 – Systems of Equations

This achievement standard involves applying systems of equations in solving problems.

| Achievement | Achievement with Merit | Achievement with Excellence |
|---|---|---|
| • Apply systems of equations in solving problems. | • Apply systems of equations using relational thinking in solving problems. | • Apply systems of equations using extended abstract thinking in solving problems. |

- This achievement standard is derived from Level 7 of The New Zealand Curriculum, Learning Media and is related to the achievement objectives
  - form and use linear and quadratic equations
  - form and use pairs of simultaneous equations, one of which may be non-linear

  in the Mathematics strand of the Mathematics and Statistics Learning Area.
- Apply systems of equations in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using appropriate representations.
- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts and terms
  - forming and using a model;

  and also relating findings to a context, or communicating thinking using appropriate mathematical statements.
- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate or solve a problem
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

  and also using correct mathematical statements, or communicating mathematical insight.
- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.
- Methods include a selection from those related to
  - forming and using a pair of simultaneous equations, one of which is non-linear
  - forming and using a system of linear inequations
  - connecting different representations of equations or inequations
  - interpreting solutions of a system of equations or inequations in context.

# Forming Equations

## Forming Equations from Word Problems

Questions are often stated in English and we need to express them as algebra in order to use our algebraic techniques to solve them.

When you examine a question such as

*John has noticed that the ages of Tahi and Sarah add to 33. Tahi is five years older than Sarah. Find the ages of Tahi and Sarah.*

You start by assigning a variable to unknowns you need to calculate. Choose a letter that makes sense in the context of the question and state what you are doing.

Let Tahi's age be $t$ and

Sarah's age be $s$

Always have a statement on what variables you are using for what, as it prevents you making errors and enables someone that is looking at your work to understand what you are doing.

Now we can read back the question replacing references to Tahi's age and Sarah's age with the letters that represent them.

$t + s$ add to 33

or better $t + s = 33$

also $t - 5 = s$

### Example

The number of tickets for adults and children at a concert is limited to 312.

Express the number of adults and children as an algebraic equation.

Let the number of adults be A

Let the number of children be C

Therefore $A + C = 312$

### Example

Express the following as an algebraic equation.
Two consecutive numbers add to 175.

Let the first number be N

Let the second be N + 1

$N + (N + 1) = 175$

**After you have decided on your variables, reread the question using the variables in the question without trying to solve the problem.**

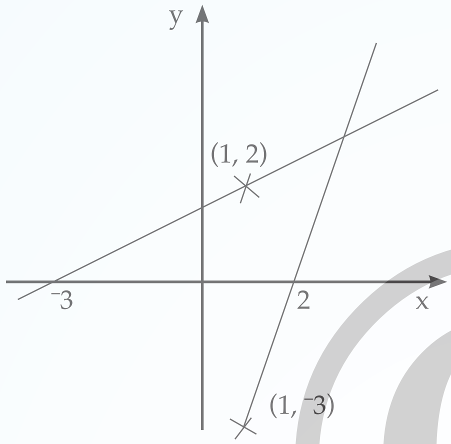
**Mathematical language has precise meaning while everyday English does not.**

**Interpret the following words and phrases as shown:**

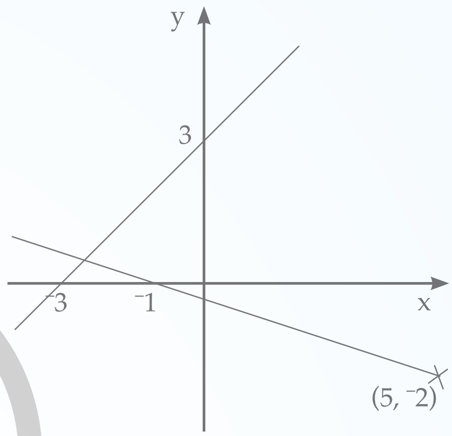
| | |
|---|---|
| *Consecutive* | **means one after each other, so algebraically the numbers would be n and n + 1.** |
| *Difference* | **means to subtract. If the difference between Claire and David's wage is \$8 this is C – D = 8.** |
| *Less than* | **means reduced or subtracted. Claire has five dollars less than David means David – \$5 = Claire.** |
| *Total* **or** *add to* | **means the numbers sum together to equal this result. A + B = result. David and Claire total \$5 means David + Claire = 5.** |
| *More than* | **means added to. Claire has \$5 more than David means Claire = David + 5.** |
| *Combined* | **means added to each other. David and Claire combined have \$5 means David + Claire = 5.** |
| *Is proportional to* | **refers to a linear or constant multiplier. If C varies proportional to D then C = kD. Claire's income is proportional to David's income means Claire = k x David.** |
| *Per, / and 'a'* | **usually mean divided by. David was paid \$18 per hour, \$18/hour or \$18 a hour.** |
| *Ratio* | **means divided by. The ratio of David's wage to Claire's wage is two means $\frac{D}{C} = 2$.** |
| **The following all refer to something being equal to:** | **is, are, was, will be, gives, yields and sold for.** |

**Merit –** A pair of lines are drawn on each of the axes.
First find the equation of each of the lines and then find the point of intersection.

**41.**

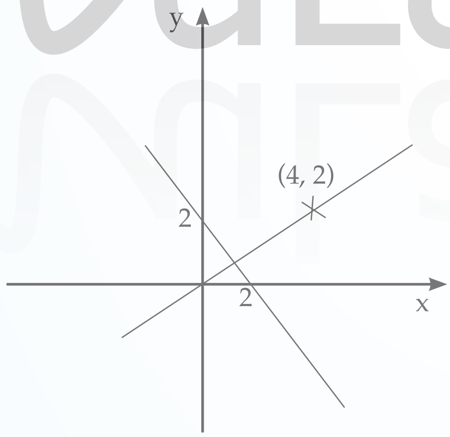



**42.**

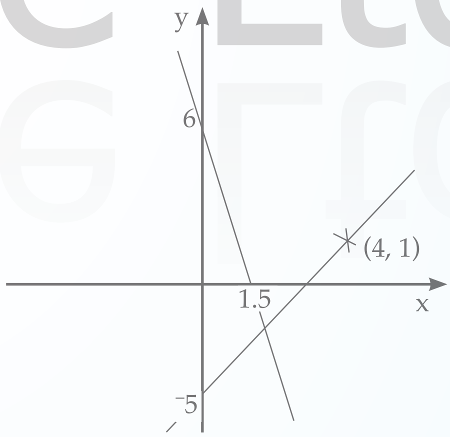





**43.**




**44.**



# Solving Linear Simultaneous Equations by Algebra

## Solving Linear Simultaneous Equations by Algebra

Using graphs with or without using a calculator is an ideal method of solving equations. It enables you to see what is going on. It also alerts you to when there are no solutions or there are multiple solutions.

This Achievement Standard requires you to understand and use a range of methods to solve simultaneous equations.

There are two common algebraic approaches. One is called Elimination and the other Substitution. Both will be explained here side-by-side but the method of elimination only works well with linear – linear problems so the authors recommend you focus on the method of substitution.

Both algebraic methods rely on each unknown in both equations being the same at the solution. Therefore we can combine both equations in such a way that there is only one unknown variable. We then solve for this one variable.

The method of elimination can be directly solved using a graphics calculator which will be explained later.

### Example

Solve the pair of simultaneous equations

$$C + 2D = 8$$
$$2C = D + 1$$

by algebra.

Using the method of Elimination the pair of equations must be in the same form. That is, each variable must be in the same relative position in each equation. In this case we will use the form $aC + bD = k$ but any form is fine as long as it is consistent. The first equation is in this form but we need to rearrange the second equation to the form $aC + bD = k$.

$$C + 2D = 8 \qquad (1)$$
$$2C - D = 1 \qquad (2)$$

We are going to combine by adding down the two equations so we want one of the variables to have the same coefficient in each equation but of opposite signs. In this case we can double the second equation so we have 2D in the first and – 2D in the second.

$$C + 2D = 8 \qquad (1)$$
$$(2) \times 2 \qquad 4C - 2D = 2 \qquad (3)$$

Now we add both equations down eliminating the variable D.

$$C + 2D = 8 \qquad (1)$$
$$4C - 2D = 2 \qquad (3)$$
$$(1) + (3) \qquad 5C + 0 = 10$$

Solving this gives $C = 2$

We substitute this value back to find D

$$2 + 2D = 8$$
$$2D = 6$$
$$D = 3$$

The simultaneous solution is $C = 2$ and $D = 3$.

### Example

Solve the pair of simultaneous equations

$$C + 2D = 8$$
$$2C = D + 1$$

by algebra.

Using the method of Substitution one (or both) of the pair of equations must be in the same form *Subject = rest of equation.* That is one equation must start C = or D =, it does not matter which.

$$C + 2D = 8 \qquad (1)$$
$$2C - D = 1 \qquad (2)$$

It is a little easier with equation (1) to make C the subject so we will use that

$$C = 8 - 2D \qquad (3)$$
$$2C - D = 1 \qquad (2)$$

Now we replace every incidence of 'C' in equation (2) with what C is equal to

$$2(8 - 2D) - D = 1 \qquad (2)$$

Simplifying we get

$$16 - 4D - D = 1$$
$${}^{-}5D = 1 - 16$$
$${}^{-}5D = {}^{-}15$$

Solving this gives $D = 3$

We substitute this value back in any equation to find C

$$C = 8 - 2 \times 3$$
$$C = 2$$

The simultaneous solution is $C = 2$ and $D = 3$.

**77.** A speculator buys two properties for a total of \$640 000. Two years later the properties are worth \$610 00 as the first property has decreased in value by 20%while the second has increased in value by 20%. How much did the first property cost originally?

**78.** To travel by taxi the 25 km from the airport to Jason's home costs \$58. When Jason takes the taxi to his office (18 km) from the airport the cost drops to \$42.60. Jason knows there is a fixed cost (flag fall) and a cost per kilometre. How much is the flag fall?

**79.** Grace goes by bike and bus the 20 km to school and it takes her half an hour. Grace bikes at 19 km/h and the bus travels at 54 km/h for the rest of the journey. How many minutes does Grace spend on the bus?

**80.** To go to a sports game the team could hire a van large enough to transport the 12 players or hire a bus and driver. It costs \$3 extra per person just to hire the bus and on top of this they pay the driver an additional \$42. The total is twice the cost of hiring the van. How much does it cost for the van?

**81.** A contractor is pricing an urgent job for a quote. It will take a worker 12 hours to complete the job.

If the worker works 8 hours at normal rate and the rest at an overtime rate the contractor will have to pay \$244.

It the worker works 5 hours at the normal rate and the rest at overtime the cost rises to \$265. How much is each rate?

**82.** John usually runs around the block he lives on. It takes him 42 minutes to complete three laps. When he is pressed for time he just runs up and back along one straight five times and this takes him 36 minutes. John runs at 200 metres a minute (12 km/h). How wide is the block John lives on?

# Algebraic Solutions to Linear and Non-linear Simultaneous Equations

## Algebraic Solutions to Non-linear Simultaneous Equations

We have already demonstrated how to graphically solve a pair of simultaneous equations one of which is non-linear. In this section we explain how this can be done using algebra.

We need to understand the algebraic approach as it is difficult to use a graphics calculator to find a solution when the non-linear equation is a circle. Also if you want to prove that there is no possible solution then you will need to use algebra.

We use the substitution method as explained for linear – linear problems. One or both of these equations needs to be in the form

Subject = equation

for this method to work.

Extra Notes

**Using a graphics calculator to solve linear and non-linear equation has its limitations.**

**It is difficult (but not impossible) to draw a circle on the graphing menu where you have a built-in solver. In this case it is easier to find the points of intersection between a straight line and a circle by algebra.**

### Example

Find the points of intersection of the line $y = x + 2$ and the hyperbola $xy = 8$.

Using the method of substitution, we begin by numbering the equations.

$$y = x + 2 \qquad (1)$$
$$xy = 8 \qquad (2)$$

Since equation (1) has y as the subject we substitute it into equation (2), i.e. wherever y appears in equation (2) we substitute $x + 2$.

| | | |
|---|---|---|
| $x(x + 2)$ | $= 8$ | subst. equ (1) into (2) |
| $x^2 + 2x$ | $= 8$ | multiplying out |
| $x^2 + 2x - 8$ | $= 0$ | simplifying |
| $(x + 4)(x - 2)$ | $= 0$ | factorising |
| $x = {}^-4$ and $x$ | $= 2$ | and solving |

To find the value of the other variables we substitute these solutions into either equation (1) or (2). We choose (1) because the algebra is simpler.

Substituting $x = {}^-4$ into (1)

$$y = {}^-4 + 2$$
$$y = {}^-2$$

Pt. of intersection $= ({}^-4, {}^-2)$

Substituting $x = 2$ into (1)

$$y = 2 + 2$$
$$y = 4$$

Pt of intersection $= (2, 4)$

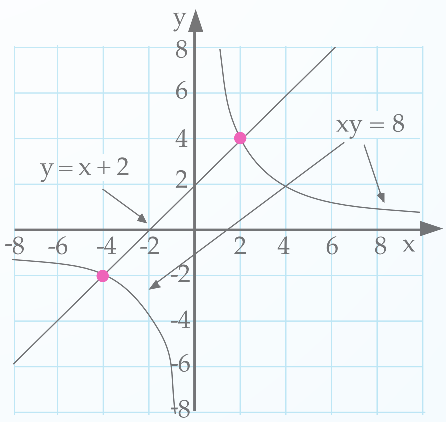
We have graphed $y = x + 2$ and $xy = 8$ on the same set of axes to illustrate these points of intersection.

**Do not use the SOLVER built into your graphics calculator.**

**The solver works well when there is one solution but unless you know approximately where other solutions are, the solver is unlikely to give you more than one solution.**

**If you want to use your graphics calculator then do so by graphing the functions and finding the points of intersection as already described.**

**Once you have reduced the problem to that of a quadratic by using substitution it is possible to solve the quadratic using the built-in quadratic solver.**



## Example

Prove there is no solution to the pair of equations

$$y = \frac{^-12}{x-1}$$
$$y = 2x + 4$$

Using the method of substitution, we begin by numbering the equations.

$$y = \frac{^-12}{x-1} \qquad (1)$$
$$y = 2x + 4 \qquad (2)$$

We substitute $(2x + 4)$ for y in equation (1).

$$2x + 4 = \frac{^-12}{x-1}$$

We multiply across by $(x – 1)$ to simplify the fraction

$$(2x + 4)(x - 1) = {^-12}$$

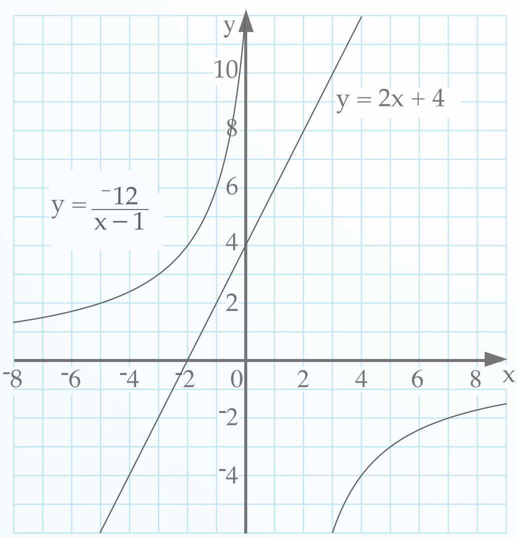
Multiplying out and putting it in the form $ax^2 + bx + c = 0$ gives

$$2x^2 + 2x - 4 + 12 = 0$$
$$2x^2 + 2x + 8 = 0$$
$$x^2 + x + 4 = 0$$

The discriminant of this equation is

$$\Delta = b^2 - 4ac$$
$$= 1^2 - 4 \times 1 \times 4$$
$$= {^-15}$$

As the $\Delta < 0$ there is no real solution. The graph is drawn below, so you can see the line and curve, but it is not part of the required solution.




## Example

Find for what values of k is the line

$$y = 2x + k$$

a tangent to the circle

$$x^2 + 2x + y^2 = 12.$$

Using the method of substitution, we begin by numbering the equations.

$$y = 2x + k \qquad (1)$$
$$x^2 + 2x + y^2 = 12 \qquad (2)$$

We substitute $2x + k$ for y in equation (2).

$$x^2 + 2x + (2x + k)^2 = 12$$

We multiply out the brackets

$$x^2 + 2x + 4x^2 + 4kx + k^2 = 12$$

As this is a quadratic we put it in the form $ax^2 + bx + c = 0$

$$5x^2 + 2x + 4kx + k^2 = 12$$
$$5x^2 + (2 + 4k)x + k^2 - 12 = 0$$

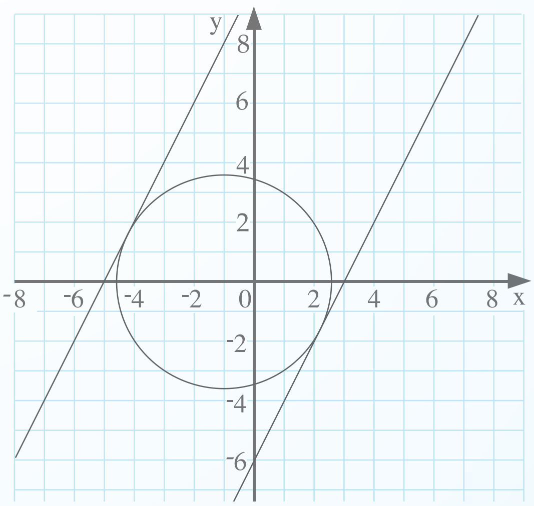
If the line is a tangent it should intersect the curve (circle) only once so the discriminant of this equation should be zero ($\Delta = 0$).

$$\Delta = b^2 - 4ac$$
$$0 = (2 + 4k)^2 - 4 \times 5 \times (k^2 - 12)$$
$$0 = 4 + 16k + 16k^2 - 20k^2 + 240$$
$$0 = 244 + 16k - 4k^2$$
$$0 = {^-4k^2} + 16k + 244$$
$$0 = k^2 - 4k - 61$$

The solution for k for this quadratic is

$$k = 10.062, {^-6.062}$$

The graph is drawn so you can see these lines are tangential to the circle.



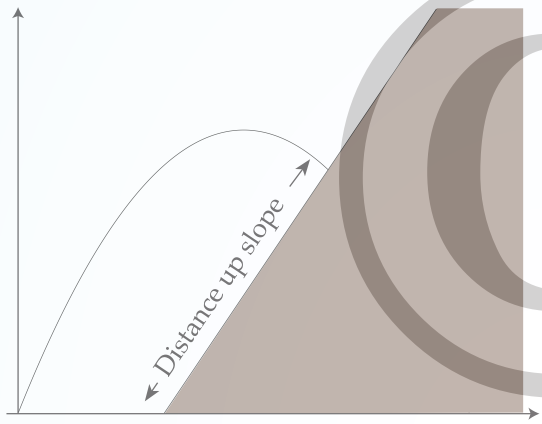
**136.** A projectile is thrown and its path is best modelled by the parabola

$$y = \frac{x(30-x)}{6}$$

It lands on a sloping bank modelled by the equation

$$y = 3x - 30$$

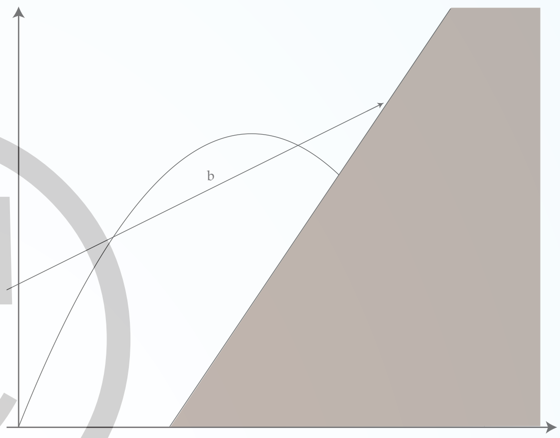
a) Find the distance up the slope that the projectile lands.




b) A high powered laser is aimed to intercept the projectile by firing along the line

$$y = x + 18 \qquad \text{(labelled b)}$$

What are the coordinates of the intersection points?



### Page 24

**71.** Let B = Budget and S = Super
B + S = 72
6B + 10S = 556
B = 41, S = 31 so
Number of drivers that select a budget wash is 41.

**72.** Let C = chip cost and
P = popcorn cost.
3C + 7P = 18
6C + 2P = 18
C = 2.5, P = 1.5 so
Cost chips = $2.50 and cost popcorn = $1.50.

**73.** Let J = Jack's age and
F = father's age.
F = J + 32
J + F = 60
J = 14, F = 46 so
Jack was born 14 years ago.

**74.** Let S = smaller angle and
L = larger angle
S + L = 180
L = 2S
S = 60, L = 120 so
larger angle is 120°.

**75.** Let D = Diana's votes and
B = Brett's votes
D + B = 125
D = B + 17
D = 71, B = 54 so
Brett got 54 votes.

**76.** Let D = Donna's fare and
M = mother's fare
D + M = 430
D = M – 40
D = 195, B = 235 so
Donna's fare cost $195.

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**77.** Let F = first property and
S = second property
F + S = 640 000
0.8F + 1.2S = 610 000
F = 395 000, S = 245 000 so
first property cost $395 000.

**78.** Let F = flag fall and
K = cost per km
F + 25K = 58
F + 18K = 42.6
F = 3, K = 2.2 so
flag fall is $3.

### Page 25 cont...

**79.** Let K = time on the bike and
B = time on the bus
K + B = 0.5
19K + 54B = 20
K = 0.2, B = 0.3 so
time on the bus is 18 minutes (0.3 hours).

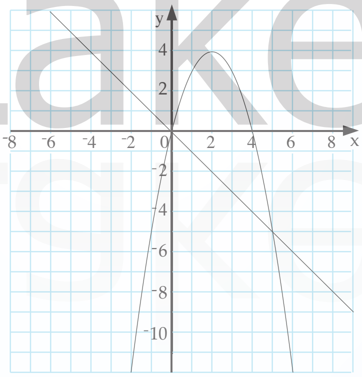
**80.** Let P = per person cost and
C = cost to hire minibus
12P = C
12(P + 3) + 42 = 2C
P = 6.5, C = 78 so
cost of the van is $78.

**81.** Let n = normal rate and
v = overtime rate
8n + 4v = 244
5n + 7v = 265
n = 18, v = 25 so
normal rate $18 and overtime rate $25.

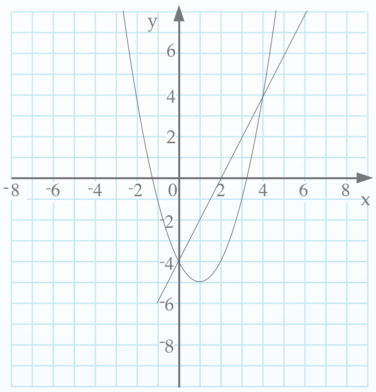
**82.** Let W = time to run width and
L = time to run length
6L + 6W = 42
10L = 36
W = 3.4, L = 3.6 so
width is 680 metres.

### Page 27

**83.** $x = 0$, $y = 0$ and $x = 5$, $y = {}^-5$.

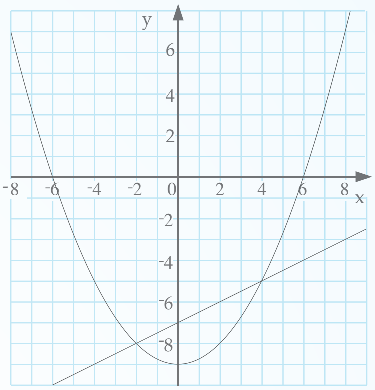


**84.** $x = 0$, $y = {}^-4$ and $x = 4$, $y = 4$.

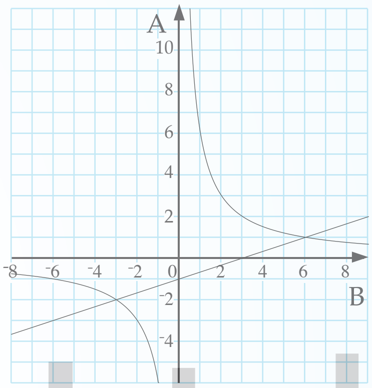


### Page 27 cont...

**85.** $x = {}^-2$, $y = {}^-8$ and $x = 4$, $y = {}^-5$.

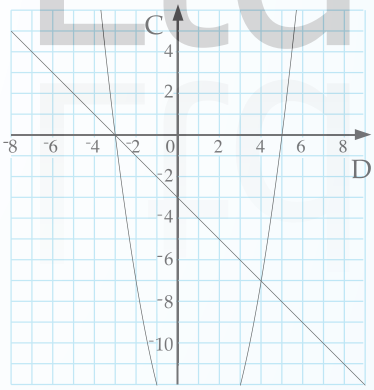


**86.** $A = 6$, $B = 1$ and $A = {}^-3$, $B = {}^-2$.



### Page 28

**87.** $D = {}^-3$, $C = 0$ and $D = 4$, $C = {}^-7$.



**88.** $F = {}^-4$, $E = 1$ and $F = 6$, $E = {}^-4$.